Rayleigh-Ritz Based Substructure Synthesis for Flexible Multibody Systems

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This paper is concerned with the modeling of flexible multibody systems by a Rayleigh-Ritz based substructure synthesis method, so that certain advantages can be accrued by using the variational approach to derive the eigenvalue problem. As with the classical Rayleigh-Ritz method, if the admissible functions used to represent the motion of the substructures are not chosen properly, convergence can suffer. This paper presents a new substructure synthesis method with superior convergence characteristics achieved by representing the motion by means of a recently developed class of functions, namely, the class of quasi-comparison functions. This improved convergence is shown to be related to improved approximation of both the differential equations and the natural boundary conditions. The theory is demonstrated by means of a numerical example.

I. Introduction

THIS investigation is concerned with the modeling of flexible multibody systems. Many structures, such as fixed-wing aircraft, helicopters, flexible spacecraft, flexible robots, etc., can be modeled as assemblages of interacting flexible bodies. A method proposed by Hurty[1,2] in the early 1960s, and known as "component-mode synthesis," consists of modeling the motion of the individual substructures, referred to as components, by means of "component modes." He uses three types of modes to represent the motion of a substructure: rigid-body modes, constraint modes, and normal modes. To define constraint modes, Hurty divides the constraints into two classes, statically determinate and redundant. The constraint modes are equal in number to the number of redundant constraints and are defined by producing a unit displacement on each redundant constraint in turn, with all other constraints fixed. The normal modes represent the modes of vibration of the components with all constraints fixed. An approach by Craig and Bampton[3] differs from that of Hurty[1,2] mainly in the selection of the component modes. Indeed, Craig and Bampton suggest two types of component modes: boundary modes providing for displacements and rotations at points along the substructure boundaries, and related to the constraint modes of Hurty, and substructure modes corresponding to completely restrained boundaries. The substructures modeled individually in Refs. 1–3 are made to act together as a single structure by eliminating the redundant generalized coordinates arising from the fact that displacements at points common to two adjacent substructures are included twice in the overall problem formulation, once for each substructure. The elimination process is based on the use of constraint equations resulting from the enforcement of compatibility conditions, which amounts to saying that the displacement of a boundary point shared by two substructures is the same. In another approach to the problem, proposed by Benfield and Hruda,[4] the effect of adjacent substructures is taken into account by subjecting a given substructure to inertial and stiffness loadings at the boundaries.

In the late 1970s and early 1980s, Meirovitch and Hale[5,6] demonstrated that the component-mode synthesis and all its variants are essentially different forms of the Rayleigh-Ritz method. Consistent with this, an approximate solution can be constructed from the space of admissible functions,[7] i.e., the functions need not be component modes. Clearly, if augmented properly, component modes can give very good results, as witnessed by the great deal of success achieved in Refs. 1–4, and in various other works expanding on these references. It should be pointed out here that the practice of using component modes has practical implications when the various components are manufactured by different organizations and the structural characteristics are provided in terms of component modes. In such cases, the component-mode synthesis can be used to integrate all this information so as to produce a structural model for the fully assembled structure. Still, however defined, component modes represent merely subspaces of the much larger space of admissible functions, and component-mode synthesis is part of a larger picture. Hence, it is more appropriate to refer to the approach as substructure synthesis.[5,6]

In the classical Rayleigh-Ritz method, a sequence of approximating solutions is constructed from the space of either admissible functions or comparison functions,[7] depending on the problem formulation (Sec. III), where admissible functions need satisfy only the geometric boundary conditions and comparison functions must satisfy all boundary conditions. However, if the problem is formulated by a variational approach, admissible functions suffice. In the case of flexible multibody systems, boundary conditions for a given substructure cannot be defined independently of the adjacent substructures, so that the use of comparison functions is not an option. Hence, the only alternative is to use merely admissible functions, which include the various "substructure modes" as special cases. This, however, raises serious questions as to the speed of convergence, which in turn is likely to require models with relatively large numbers of degrees of freedom.

Quite recently, Meirovitch and Kwak[8] demonstrated that the classical Rayleigh-Ritz method as ordinarily used has an implicit flaw, which can impair its convergence characteristics. As just pointed out, if the eigenvalue problem is formulated as a variational problem, the Rayleigh-Ritz method consists of constructing a sequence of approximate solutions from the space of admissible functions. These admissible functions can be expressed in the form of series of trial functions, and

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accuracy of the approximating sequence is improved by increasing the number of terms in the series. Because the trial functions in a given series are commonly taken as members of the same family of functions, it follows that each trial function must be an admissible function. In many cases, solutions in the form of series of admissible functions of the same type are characterized by poor convergence, which can be traced to the fact that a finite linear combination of admissible functions of the same type is not able to satisfy the natural boundary conditions, in addition to not satisfying the differential equation. To overcome this predicament, Meirovitch and Kwak[8] proposed that the approximating sequence be from the space of quasi-comparison functions, instead of merely from the space of admissible functions. The quasi-comparison functions represent a new class of functions with superior convergence characteristics. The theory developed in Ref. 8 is summarized in Sec. III, in which some additional results and insights are provided. The concepts developed in Ref. 8 for single elastic members have been extended to substructure synthesis in Ref. 9. Indeed, it is in substructure synthesis that the power and versatility of quasicomparison functions become evident.

This paper presents a substructure synthesis theory identified closely with the Rayleigh-Ritz theory and based on the concept of quasi-comparison functions. In the process, new insights into the nature of the approximate solutions to the eigenvalue problem generated by the classical Rayleigh-Ritz method are provided. In particular, the question of how well the differential equation is satisfied is explored. Examples using single elastic members and flexible multibody systems demonstrate the remarkable convergence characteristics of the proposed method. The example involving the multibody system demonstrates in a heuristic way the nonuniqueness of the approximate solution. Indeed, different sets of quasi-comparison functions are capable of producing equally good results. Consistent with the nonuniqueness idea, it can be expected that various component-mode synthesis methods will yield similarly good results, provided the "modes" used are from the space of quasi-comparison functions.

II. Differential Eigenvalue Problem and Stationarity of Rayleigh's Quotient

Distributed elastic structures are characterized by differential eigenvalue problems, which can be described in general form by the differential equation[7]

$$\mathcal{L}W(x) = \lambda m(x)W(x), \qquad 0<x<L \tag{1}$$

where $\mathcal{L}$ is a linear self-adjoint differential operator of order $2p$, in which p is an integer, W is the displacement of a given point in the nominal position x, λ is a parameter, $m(x)$ is the mass density at x, and L is the length of the structure. The displacement is subject to the boundary conditions

$$B_iW = 0, \qquad i = 1,2, \ldots, p, \qquad x = 0,L \tag{2}$$

where B_i are linear homogenous differential operators of maximum order $2p-1$. The solution of the eigenvalue problem, Eqs. (1) and (2), consists of a denumerably infinite set of eigenvalues λ_r and eigenfunctions $W_r(x)(r = 1,2,\ldots)$.

The eigenfunctions are orthogonal and can be normalized so as to satisfy the orthonormality relations

$$\int_0^L mW_rW_s\,dx = \delta_{rs}, \quad \int_0^L W_r\mathcal{L}W_s\,dx = \lambda_r\delta_{rs}, \quad r,s = 1,2,\ldots \tag{3}$$

where δ_{rs} is the Kronecker delta. The eigenfunctions form a complete set, which implies that an arbitrary function can be approximated to any degree of accuracy by a linear combination of the eigenfunctions by simply increasing the number of terms in the linear combination. This permits us to formulate the expansion theorem, which can be stated as follows: Every function w with continuous $\mathcal{L}w$ and satisfying the boundary conditions of the system can be expanded in the absolutely and uniformly convergent series of eigenfunctions

$$w = \sum_{r=1}^{\infty} c_rW_r \tag{4}$$

where the coefficients are given by

$$c_r = \int_0^L mwW_r\,dx, \qquad r = 1,2,\ldots \tag{5}$$

Next, let us multiply Eq. (1) by $W(x)$, integrate over the domain of the structure and rearrange to obtain

$$R(W) = \lambda = \frac{\int_0^L W\mathcal{L}W\,dx}{\int_0^L mW^2\,dx} \tag{6}$$

which is known as Rayleigh's quotient. Using the expansion theorem, Eqs. (4) and (5), it can be shown[7] that Rayleigh's quotient has a stationary value in the neighborhood of an eigenfunction, where the stationary value is the associated eigenvalue. Hence, finding the stationary values of Rayleigh's quotient is equivalent to solving the differential eigenvalue problem, Eqs. (1) and (2). It turns out that, in seeking approximate solutions to the eigenvalue problem, it is often preferable to render Rayleigh's quotient stationary rather than to solve a differential eigenvalue problem.

III. Rayleigh-Ritz Method

More often than not, it is not possible to solve the eigenvalue problem for distributed structures in closed form. The Rayleigh-Ritz method is a technique for obtaining approximate solutions to the eigenvalue problem. Before discussing the Rayleigh-Ritz method, it is advisable to introduce some additional classes of functions.

Eigenfunctions satisfy both the differential equation, Eq. (1), and the boundary conditions, Eq. (2). Clearly, they are $2p$ times differentiable. Except for some simple cases, almost invariably characterized by constant coefficients in the differential equation and simple boundary conditions, the eigenvalue problem does not admit closed-form solutions, so that exact eigenfunctions are beyond reach. In seeking approximate solutions to the eigenvalue problem, it is obvious that something must be violated. To this end, we no longer insist that the solutions satisfy the differential equation, or that they render Rayleigh's quotient stationary, although we continue to insist that they satisfy all the boundary conditions of the system. Because both the differential equation, Eq. (1), and Rayleigh's quotient, Eq. (6), involve $\mathcal{L}W$, the approximate solutions must be differentiable $2p$ times. Functions that are differentiable $2p$ times and satisfy all the boundary conditions are known as comparison functions.[7] One approach consists of constructing approximate solutions from the space of comparison functions.

According to the Rayleigh-Ritz method, a sequence of approximating solutions to the eigenvalue problem is assumed in the form

$$W^n = \sum_{i=1}^{n} u_i\phi_i = \boldsymbol{\phi}^T\boldsymbol{u}, \qquad n = 1,2,\ldots \tag{7}$$

where u_i are undetermined coefficients and ϕ_i are trial functions, and $\boldsymbol{u}$ and $\boldsymbol{\phi}$ are the corresponding vectors. The solutions W^n must be from the space of comparison functions, from which it follows that the trial functions must be comparison functions. The comparison functions must be from a complete set, which implies that the solution to the differential eigenvalue problem can be made as accurate as desired by merely increasing the number n of comparison functions in

series (7). Inserting Eq. (7) into Eq. (1), or Eq. (6), and following the usual steps,[7] we obtain the algebraic eigenvalue problem

$$\sum_{j=1}^{n} k_{ij}u_j = \lambda^n \sum_{j=1}^{n} m_{ij}u_j, \qquad i = 1,2, \ldots, n \tag{8}$$

where the superscript n indicates the number of terms in the series and in which

$$k_{ij} = k_{ji} = \int_0^L \phi_i \mathcal{L}\phi_j \, dx, \qquad i,j = 1,2, \ldots, n \tag{9}$$

are symmetric stiffness coefficients and

$$m_{ij} = m_{ji} = \int_0^L m\phi_i\phi_j \, dx, \qquad i,j = 1,2, \ldots, n \tag{10}$$

are symmetric mass coefficients. Note that the symmetry of the stiffness coefficients is guaranteed by the self-adjointness of $\mathcal{L}$. The solution of Eqs. (8) yields the approximate eigenvalues $\lambda_r^n (r = 1,2, \ldots, n)$ and the eigenvectors u_r. The approximate eigenfunctions W_r^n are obtained by inserting the eigenvectors u_r into Eq. (7).

The fact that the Rayleigh-Ritz method just described requires comparison functions to construct approximate solutions to the differential eigenvalue problem proves at times to be a serious drawback, as comparison functions are not easy to generate. In view of this, we wish to consider another version of the Rayleigh-Ritz method. To this end, we distinguish between geometric and natural boundary conditions. The geometric boundary conditions involve the displacement or the slope at the boundaries and are characterized by operators B_i of maximum order $p - 1$. On the other hand, natural boundary conditions involve moment or shearing force balance at the boundaries and are characterized by operators B_i of maximum order $2p - 1$. It is the satisfaction of the natural boundary conditions that proves troublesome at times. To obviate this problem, we integrate the numerator of Rayleigh's quotient by parts, taking into account the boundary conditions. The result can be written in the form

$$\int_0^L W\mathcal{L}W \, dx = \int_0^L \sum_{k=0}^{p} a_k \left(\frac{d^k W}{dx^k}\right)^2 dx + \sum_{l=0}^{p-1} b_l \left(\frac{d^l W}{dx^l}\right)^2 \Bigg|_0^L \tag{11}$$

where $a_k (k = 0,1,2, \ldots, p)$ and $b_l (l = 0,1,2, \ldots, p - 1)$ are coefficients depending in general on x. The expression

$$[W,W] = \int_0^L \sum_{k=0}^{p} a_k \left(\frac{d^k W}{dx^k}\right)^2 dx + \sum_{l=0}^{p-1} b_l \left(\frac{d^l W}{dx^l}\right)^2 \Bigg|_0^L \tag{12}$$

is known as an energy inner product.[7] The name can be easily explained by pointing out that the expression is proportional to the potential energy in free vibration. Considering Eqs. (11) and (12), we can rewrite the Rayleigh quotient in the form

$$R(W) = \lambda = [W,W] \Big/ \int_0^L mW^2 \, dx \tag{13}$$

Then, examining the energy inner product at the numerator, we observe that the integrand involves derivatives of W of maximum order p. Moreover, the natural boundary conditions are accounted for in the boundary terms, which involve derivatives of W of maximum order $p - 1$. Hence, the energy inner product is defined for functions that are only p times differentiable and satisfy the geometric boundary conditions alone. Such functions are known as admissible functions. It follows that in using the alternative form of Rayleigh's quotient, Eq. (13), it is possible to construct an approximate solution to the eigenvalue problem in the form of Eq. (7), where now W^n is from the space of admissible functions, which implies that the functions ϕ_i are only admissible functions, instead of comparison functions. The admissible functions must be from a set that is complete in energy, which implies that the energy inner product can be made as accurate as desired by merely increasing the number n of admissible functions in series (7). It is shown in Ref. 7 that the use of Eq. (13) to render Rayleigh's quotient stationary is equivalent to solving the differential eigenvalue problem, Eq. (1). We refer to this approach as the variational approach to the eigenvalue problem. To obtain the approximate solution to the eigenvalue problem, we must once again solve the eigenvalue problem given by Eqs. (8), but this time the stiffness coefficients are given by

$$k_{ij} = k_{ji} = \int_0^L \sum_{k=0}^{p} a_k \frac{d^k\phi_i}{dx^k}\frac{d^k\phi_j}{dx^k} \, dx + \sum_{l=0}^{p-1} b_l \frac{d^l\phi_i}{dx^l}\frac{d^l\phi_j}{dx^l} \Bigg|_0^L$$

$$i,j = 1,2, \ldots, n \tag{14}$$

instead of Eqs. (9). Clearly, the mass coefficients remain in the form of Eqs. (10).

IV. Convergence Enhancement via Quasi-Comparison Functions

In Sec. III, it was demonstrated that, in constructing approximate solutions to the eigenvalue problem by the Rayleigh-Ritz method, some advantage can be derived from using the variational approach based on the form of Rayleigh's quotient in terms of the energy inner product, Eq. (13). The advantage is that the energy inner product requires only admissible functions, which tend to be simpler than comparison functions and are considerably more abundant. On the other hand, in some cases convergence can suffer, because the number of admissible functions required for a certain degree of accuracy tends to be significantly larger than the number of comparison functions. This can be traced to the fact that natural boundary conditions are often very difficult to satisfy with a relatively small number of admissible functions. Whereas the Rayleigh-Ritz theory guarantees convergence, provided the admissible functions are from a complete set, this is small comfort in computational work, in which the object is to obtain good accuracy with a number of terms as small as possible. Hence, the question is how to retain the advantage of the variational approach to the eigenvalue problem without sacrificing accuracy.

The resolution of the preceding dilemma was provided in Ref. 8, in which it was demonstrated that a flaw exists in the Rayleigh-Ritz practice, as long as the trial functions are of one type only. Indeed, in defining the various classes of functions, the requirements are placed on every number of the class individually. However, in constructing an approximate solution, the functions are never taken individually but in linear combinations. In recognition of this fact, a new class of functions was created in Ref. 8, namely, the class of quasi-comparison functions. The quasi-comparison functions are defined as linear combinations of admissible functions that act like comparison functions. This implies that a minimum number of admissible functions is required before the linear combination becomes capable of satisfying all the boundary conditions of the system, including the natural boundary conditions. Clearly, individually, the admissible functions are not able to satisfy the natural boundary conditions; otherwise, they would be comparison functions. The general idea is to select admissible functions so as to provide the possibility of satisfying the natural boundary conditions. This requires that *the solution W^n be constructed from several different types of admissible functions.* It is this variety of admissible functions that permits accurate satisfaction of the natural boundary conditions with a relatively small number of terms; such a feat is impossible with admissible functions of a single type. It should be pointed out here that, although it is possible to

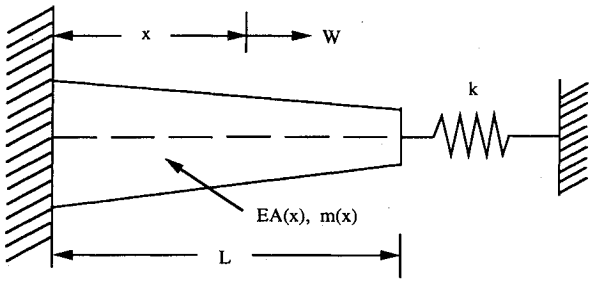

Fig. 1 Tapered rod in axial vibration.

adjust the undetermined coefficients u_i so as to satisfy the natural boundary conditions exactly, this is neither necessary nor even desirable. Indeed, the coefficients u_i are determined as a result of rendering the Rayleigh quotient stationary, which amounts to solving the algebraic eigenvalue problem, Eqs. (8), with the mass coefficients in the form of Eqs. (10) and the stiffness coefficients in the form of Eqs. (14). But, because the Rayleigh quotient involves quantities defined over both the interior and boundaries of the structure, it follows that the Rayleigh-Ritz process adjusts the undetermined coefficients so that both the differential equation and the natural boundary conditions are satisfied as well as possible. As pointed out below, a solution in terms of quasi-comparison functions can lead to more accurate satisfaction of the differential equation than a solution in terms of comparison functions alone.

Reference 8 includes a numerical example in which the convergence characteristics of quasi-comparison functions are shown to be indistinguishable from those of comparison functions and far superior to those of ordinary admissible functions, as far as the eigenvalues are concerned. The example of Ref. 8 involves the tapered rod in axial vibration shown in Fig. 1, which is characterized by a geometric boundary condition at $x = 0$ and a natural boundary condition at $x = L$. The system parameters are given by

$$EA(x) = (6EA/5)[1 - \tfrac{1}{2}(x/L)^2] \tag{15a}$$

$$k = EA/L \tag{15b}$$

$$m(x) = (6m/5)[1 - \tfrac{1}{2}(x/L)^2] \tag{15c}$$

where $EA(x)$ is the axial stiffness, k the stiffness of the spring at $x = L$, and $m(x)$ the mass density. The differential operator $\mathcal{L}$ has the form

$$\mathcal{L} = -\frac{d}{dx}\left[EA(x)\frac{d}{dx}\right], \qquad 0<x<L \tag{16}$$

and the boundary differential operators are

$$B_1 = 1 \qquad \text{at } x = 0 \tag{17a}$$

$$B_1 = EA\frac{d}{dx} + k \qquad \text{at } x = L \tag{17b}$$

leading to the energy inner product

$$[W,W] = \int_0^L EA(x)\left[\frac{dW(x)}{dx}\right]^2 dx + kW^2(L) \tag{18}$$

The eigenvalue problem was solved using the ordinary admissible functions (AF)

$$\phi_i = \sin\{[(2i-1)\pi x]/2L\}, \qquad i = 1,2, \ldots, n \tag{19}$$

which are recognized as the eigenfunctions of a uniform clamped-free rod and quasi-comparison functions (QCF)

Table 1 First natural frequency

n	AF	QCF	CF
1	0.37078	0.37078	0.35380
2	0.36175	0.35390	0.35276
3	0.35866	0.35271	0.35264
4	0.35709	0.35262	0.35262
5	0.35616	0.35261	0.35262
6	0.35554	—	0.35261
7	0.35510	—	—
8	0.35477	—	—
9	0.35452	—	—
10	0.35432	—	—

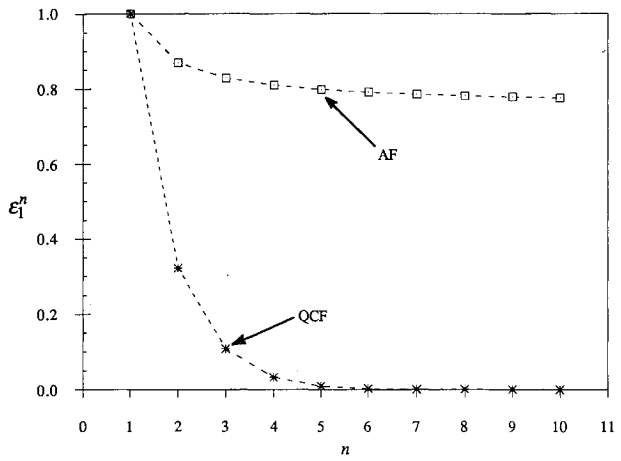

Fig. 2 Error in the natural boundary condition vs the number of terms.

from the set

$$\phi_i = \sin(i\pi x/2L), \qquad i = 1,2, \ldots, n \tag{20}$$

which represent the eigenfunctions of a uniform clamped-free rod for i odd and clamped-clamped rod for i even, and the comparison functions (CF)

$$\phi_i = \sin\beta_i x, \qquad i = 1,2, \ldots, n \tag{21}$$

where β_i are obtained by solving the transcendental equation

$$EA(L)\beta\cos\beta L + k\sin\beta L = 0 \tag{22}$$

Clearly, the quasi-comparison functions constructed from the set given by Eq. (20) represent a combination of two types of admissible functions. The first provides for nonzero displacement at $x = L$ and the second provides for nonzero slope at $x = L$. Hence, a small linear combination of these two types of admissible functions is capable of satisfying all the boundary conditions. The comparison functions given by Eq. (21) are recognized as the eigenfunctions of a uniform rod clamped at $x = 0$ and with a spring of stiffness k at $x = L$, and we note that Eq. (22) arises from the satisfaction of the natural boundary condition at $x = L$. The first computed natural frequency for various values of n are shown in Table 1 for the three sets of functions. The results provide clear demonstration of the excellent convergence characteristics of quasi-comparison functions.

As a matter of interest, we wish to verify how well the solution in terms of the quasi-comparison functions from the set given by Eq. (20) satisfies the natural boundary conditions. To this end, we refer to Eq. (17b) and define the error in the boundary condition at $x = L$ arising from using an n-term approximate solution as follows:

$$\epsilon_1^n(L) = EA\frac{dW^n}{dx}\bigg|_{x=L} + kW^n(L) \tag{23}$$

where, as in Eq. (7), the superscript n refers to the number of terms in the series. Figure 2 shows a plot of ϵ_1^n vs n corresponding to the first computed mode. From the figure, it can be seen that the natural boundary condition is virtually satisfied with quasi-comparison functions consisting of six terms, $n = 6$. For comparison purpose, Fig. 2 shows also the plot corresponding to the ordinary admissible functions given by Eqs. (19). It is clear that the error is relatively large and, very disturbingly, it goes down at a painfully slow rate, so that the admissible functions given by Eqs. (19) represent a poor choice.

The satisfaction of the natural boundary conditions is highly desirable in itself, but it has broader implications. Indeed, we observe from Eqs. (12) and (13) that the boundary points are no more privileged than the interior points. Hence, in using Eq. (13) to render Rayleigh's quotient stationary, it is reasonable to expect that the solution thus produced is accurate not only at the boundaries, but at all other points of the structure as well. To verify this conjecture, we define the error in the differential equation at any interior point as follows:

$$\epsilon_r^n(x) = \mathcal{L}W_r^n(x) - \lambda_r^n m(x) W_r^n(x), \qquad r = 1,2, \ldots, n \tag{24}$$

Table 2 shows values of $\epsilon_1^{10}(x)$ at various interior points x, corresponding to the first computed mode for all three sets of functions, Eqs. (19-21), and Fig. 3 shows the norm $\|\epsilon_1^n(x)\|$ for $n = 1,2, \ldots, 10$. The results justify the expectation fully, as the solution in terms of quasi-comparison functions is considerably more accurate than that in terms of ordinary admissible functions. In fact, the solution in terms of admissible functions tends to become less and less accurate as n increases. Note that the solution in terms of quasi-comparison functions is even more accurate than that in terms of comparison functions. This can be attributed to the fact that the larger variety obtained by combining two types of admissible functions permits a less constrained minimization process, thus permitting a lower minimum. This results in a more accurate solution throughout the domain of the structure, except at $x = L$, where the comparison functions satisfy the natural boundary conditions exactly. Even at $x = L$, the solution in terms of quasi-comparison functions is very accurate for $n > 6$ (Fig. 2). We recall that there is a minimal number of terms before the linear combinations become quasi-comparison functions, so that the curve QCF is not meaningful for small n.

V. Eigenvalue Problem for Flexible Multibody Systems

In this section, we propose to derive first the free-vibration equations of motion for flexible multibody systems of the type shown in Fig. 4. Then, the eigenvalue problem follows immediately from the free-vibration equations. Because the eigenvalue problem represents a linear problem, the interest lies in linear equations of motion, which implies that all displacements must be small. We derive the equations of motion by means of Lagrange's equations, which in the case of free vibration of undamped systems amounts to deriving the kinetic energy and potential energy.[7] The latter two are fully defined by the mass matrix and stiffness matrix, respectively.

Table 2 Error in the differential equation

x/L	AF	QCF	CF
0.05	−1.45031	0.00174	0.02744
0.10	0.03897	0.00178	−0.00130
0.15	1.47816	0.00005	−0.02791
0.20	−0.07963	−0.00175	0.00268
0.25	−1.53726	−0.00181	0.02890
0.30	0.12391	−0.00004	−0.00425
0.35	1.63554	0.00182	−0.03053
0.40	−0.17432	0.00182	0.00615
0.45	−1.78854	−0.00013	0.03303
0.50	0.23454	−0.00201	−0.00862
0.55	2.02704	−0.00172	−0.03681
0.60	−0.31069	0.00061	0.01218
0.65	−2.41796	0.00235	0.04272
0.70	0.41443	0.00120	−0.01798
0.75	3.13535	−0.00172	−0.05263
0.80	−0.57301	−0.00239	0.02957
0.85	−4.81343	0.00091	0.07139
0.90	0.87772	0.00305	−0.06313
0.95	13.10726	−0.00222	−0.11062

The task of deriving the equations of motion can be simplified appreciably by adopting a consistent kinematical procedure for describing the motion. To this end, we introduce an inertial set of axes $X_I Y_I Z_I$ with the origin at I, a set of body axes $x_o y_o z_o$ with the origin at O and attached to substructure o in the undeformed state, a set of body axes $x_a y_a z_a$ with the origin at A and attached to substructure a in the undeformed state, etc. The various axes are shown in Fig. 4. For simplicity, we limit the formulation to substructures of the type o and a. Extension of the formulation to substructures of the type b, c, etc., is obvious, but tends to be exceedingly laborious. To carry out the extension, we observe that the motion of b relates to the motion of a in the same manner as the motion of a relates to the motion of o. The position vector of typical points in o and a can be written as $\boldsymbol{R}_o = \boldsymbol{R}_O + \boldsymbol{r}_o + \boldsymbol{w}_o$ and $\boldsymbol{R}_a = \boldsymbol{R}_O + \boldsymbol{r}_{oa} + \boldsymbol{w}_{oa} + \boldsymbol{r}_a + \boldsymbol{w}_a$ ($a = 1,2, \ldots, N$), respectively, where $\boldsymbol{R}_O$ is the radius vector from I to O, $\boldsymbol{r}_o$ the radius vector from O to a typical point in o, $\boldsymbol{w}_o$ the elastic displacement vector of the same point measured relative to axes $x_o y_o z_o$, $\boldsymbol{r}_{oa}$ the radius vector from O to A, $\boldsymbol{w}_{oa}$ the vector $\boldsymbol{w}_o$ evaluated at A, $\boldsymbol{r}_a$ the radius vector from A to a typical point in a, and $\boldsymbol{w}_a$ the elastic displacement of the same point measured relative to axes $x_a y_a z_a$. Note that all vectors are in terms of components along local axes.

The Lagrangian formulation requires the kinetic energy, which in turn requires the velocity of typical points in the various substructures. To derive expressions for these velocities, we assume that axes $x_o y_o z_o$ rotate with the angular velocity $\dot{\boldsymbol{\theta}}$ relative to the inertial space and that axes $x_a y_a z_a$ rotate with the angular velocity $\dot{\boldsymbol{\beta}}_a$ relative to $x_o y_o z_o$, due to the elastic motion at a. Recalling that we are interested in linearized equations of motion, the velocity vector for a typical point in substructure o and substructures a can be written as follows:

$$\dot{\boldsymbol{R}}_o = \dot{\boldsymbol{R}}_O + \tilde{\boldsymbol{r}}_o^T \dot{\boldsymbol{\theta}} + \dot{\boldsymbol{w}}_o \tag{25a}$$

$$\dot{\boldsymbol{R}}_a = \dot{\boldsymbol{R}}_O + (\tilde{\boldsymbol{r}}_{oa}^T + \boldsymbol{C}_a^T \tilde{\boldsymbol{r}}_a^T \boldsymbol{C}_a)\dot{\boldsymbol{\theta}} + \dot{\boldsymbol{w}}_{oa} + \boldsymbol{C}_a^T \tilde{\boldsymbol{r}}_a^T \boldsymbol{C}_a \dot{\boldsymbol{\beta}}_a + \boldsymbol{C}_a^T \dot{\boldsymbol{w}}_a$$

$$a = 1,2, \ldots, N \tag{25b}$$

where $\dot{\boldsymbol{R}}_O$ is the velocity vector of O, and $\boldsymbol{C}_a$ is a matrix of direction cosines between $x_a y_a z_a$ and $x_o y_o z_o$. Moreover, $\boldsymbol{\beta}_a = \nabla \times \boldsymbol{w}_{oa}$, and we note that, in writing the angular dis-

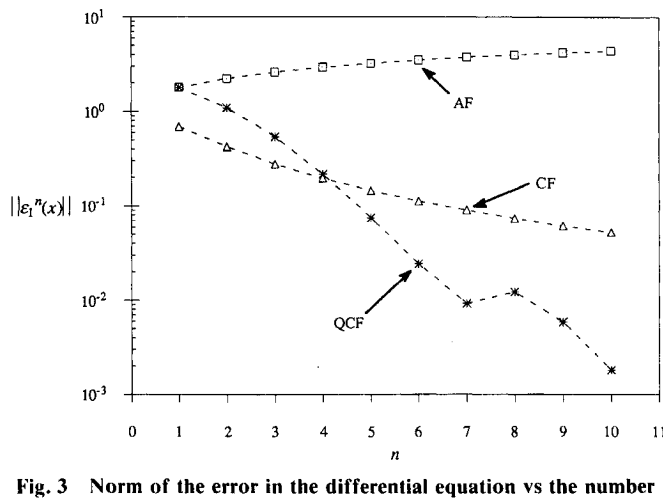

Fig. 3 Norm of the error in the differential equation vs the number of terms.

placements due to elastic deformations in vector form, we take into account that these deformations are small. We also note that a tilde over a symbol denotes a skew symmetric matrix formed from the corresponding vector.[10]

The degrees of freedom of the system are associated with the rigid-body motions of the frame $x_o y_o z_o$ and the elastic motions of the substructures. In view of this, the equations of motion will be hybrid, i.e., ordinary differential equations for the rigid-body motions and partial differential equations for the elastic motions. The object of this paper is to study certain implications of modeling by the Rayleigh-Ritz method. To this end, we discretize the partial differential equations by assuming that the elastic displacements can be expressed in the form

$$w_s(r_s,t) = \Phi_s(r_s)q_s(t), \quad s = o,a, \quad a = 1,2, \ldots, N \tag{26}$$

where Φ_s are matrices of admissible functions and q_s are vectors of generalized displacements. Using Eqs. (25) in conjunction with Eqs. (26), the kinetic energy can be reduced to the form

$$\begin{aligned} T &= \frac{1}{2}\int_{D_o} \rho_o \dot{\boldsymbol{R}}_o^T \dot{\boldsymbol{R}}_o \, dD_o + \frac{1}{2}\sum_{a=1}^{N}\int_{D_a} \rho_a \dot{\boldsymbol{R}}_a^T \dot{\boldsymbol{R}}_a \, dD_a \\ &\cong \frac{1}{2} m_t \boldsymbol{V}_O^T \boldsymbol{V}_O - \boldsymbol{V}_O^T \tilde{\boldsymbol{S}}_t \boldsymbol{\omega} + \boldsymbol{V}_O^T \tilde{\boldsymbol{\Phi}}_t \dot{\boldsymbol{q}}_o + \boldsymbol{V}_O^T \sum_{a=1}^{N} \boldsymbol{C}_a^T \bar{\boldsymbol{\Phi}}_a \dot{\boldsymbol{q}}_o \\ &\quad + \frac{1}{2}\boldsymbol{\omega}^T \boldsymbol{I}_t \boldsymbol{\omega} + \boldsymbol{\omega}^T \tilde{\boldsymbol{\Phi}}_t \dot{\boldsymbol{q}}_o + \boldsymbol{\omega}^T \sum_{a=1}^{N} \boldsymbol{H}_a \dot{\boldsymbol{q}}_a + \frac{1}{2}\dot{\boldsymbol{q}}_o^T \boldsymbol{M}_t \dot{\boldsymbol{q}}_o \\ &\quad + \dot{\boldsymbol{q}}_o^T \sum_{a=1}^{N} \boldsymbol{J}_a \dot{\boldsymbol{q}}_a + \frac{1}{2}\sum_{a=1}^{N} \dot{\boldsymbol{q}}_a^T \boldsymbol{M}_a \dot{\boldsymbol{q}}_a = \frac{1}{2}\dot{\boldsymbol{x}}^T \boldsymbol{M} \dot{\boldsymbol{x}} \end{aligned} \tag{27}$$

where $\boldsymbol{x} = [\boldsymbol{R}_O^T \boldsymbol{\theta}^T \boldsymbol{q}_o^T \boldsymbol{q}_1^T \boldsymbol{q}_2^T \cdots \boldsymbol{q}_N^T]^T$ is the configuration vector, in which we note that $\dot{\boldsymbol{R}}_o = \boldsymbol{V}_o$ and $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}$, and

$$\boldsymbol{M} = \begin{bmatrix} m_t & \tilde{\boldsymbol{S}}_t^T & \bar{\boldsymbol{\Phi}}_t & \boldsymbol{C}_1^T\bar{\boldsymbol{\Phi}}_1 & \boldsymbol{C}_2^T\bar{\boldsymbol{\Phi}}_2 & \cdots & \boldsymbol{C}_N^T\bar{\boldsymbol{\Phi}}_N \\ \tilde{\boldsymbol{S}}_t & \boldsymbol{I}_t & \tilde{\boldsymbol{\Phi}}_t & \boldsymbol{H}_1 & \boldsymbol{H}_2 & \cdots & \boldsymbol{H}_N \\ \bar{\boldsymbol{\Phi}}_t^T & \tilde{\boldsymbol{\Phi}}_t^T & \boldsymbol{M}_t & \boldsymbol{J}_1 & \boldsymbol{J}_2 & \cdots & \boldsymbol{J}_N \\ \bar{\boldsymbol{\Phi}}_1^T\boldsymbol{C}_1 & \boldsymbol{H}_1^T & \boldsymbol{J}_1^T & \boldsymbol{M}_1 & 0 & \cdots & 0 \\ \bar{\boldsymbol{\Phi}}_2^T\boldsymbol{C}_2 & \boldsymbol{H}_2^T & \boldsymbol{J}_2^T & 0 & \boldsymbol{M}_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \bar{\boldsymbol{\Phi}}_N^T\boldsymbol{C}_N & \boldsymbol{H}_N^T & \boldsymbol{J}_N^T & 0 & 0 & \cdots & \boldsymbol{M}_N \end{bmatrix} \tag{28}$$

is the mass matrix. The various quantities entering into Eq. (28) are as follows:

$$\begin{aligned} m_t &= m_o + \sum_{a=1}^{N} m_a, \qquad \tilde{\boldsymbol{S}}_t = \tilde{\boldsymbol{S}}_o + \sum_{a=1}^{N} (m_a \tilde{\boldsymbol{r}}_{oa} + \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a) \\ \bar{\boldsymbol{\Phi}}_t &= \bar{\boldsymbol{\Phi}}_o + \sum_{a=1}^{N} (m_a \boldsymbol{\Phi}_{oa} - \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a \boldsymbol{\Upsilon}_{oa}) \\ \boldsymbol{I}_t &= \boldsymbol{I}_o + \sum_{a=1}^{N} (\boldsymbol{C}_a^T \boldsymbol{I}_a \boldsymbol{C}_a - m_a \tilde{\boldsymbol{r}}_{oa}^2 - \tilde{\boldsymbol{r}}_{oa} \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a - \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a \tilde{\boldsymbol{r}}_{oa}) \\ \tilde{\boldsymbol{\Phi}}_t &= \tilde{\boldsymbol{\Phi}}_o + \sum_{a=1}^{N} [(m_a \tilde{\boldsymbol{r}}_{oa} + \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a)\boldsymbol{\Phi}_{oa} \\ &\quad + (\boldsymbol{C}_a^T \boldsymbol{I}_a \boldsymbol{C}_a - \tilde{\boldsymbol{r}}_{oa} \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a)\boldsymbol{\Upsilon}_{oa}] \\ \boldsymbol{M}_t &= \boldsymbol{M}_o + \sum_{a=1}^{N} (m_a \boldsymbol{\Phi}_{oa}^T \boldsymbol{\Phi}_{oa} - \boldsymbol{\Phi}_{oa}^T \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a \boldsymbol{\Upsilon}_{oa} \\ &\quad + \boldsymbol{\Upsilon}_{oa}^T \boldsymbol{C}_a^T \tilde{\boldsymbol{S}}_a \boldsymbol{C}_a \boldsymbol{\Phi}_{oa} + \boldsymbol{\Upsilon}_{oa}^T \boldsymbol{C}_a^T \boldsymbol{I}_a \boldsymbol{C}_a \boldsymbol{\Upsilon}_{oa}) \\ \boldsymbol{H}_s &= \boldsymbol{C}_s^T \bar{\boldsymbol{\Phi}}_s + \tilde{\boldsymbol{r}}_{os} \boldsymbol{C}_s^T \bar{\boldsymbol{\Phi}}_s, \qquad \boldsymbol{J}_s = \boldsymbol{\Gamma}_s^T \boldsymbol{C}_s^T \bar{\boldsymbol{\Phi}}_s + \boldsymbol{\Phi}_{os}^T \boldsymbol{C}_s^T \tilde{\boldsymbol{\Phi}}_s \\ \boldsymbol{M}_s &= \int_{D_s} \rho_s \boldsymbol{\Phi}_s^T \boldsymbol{\Phi}_s \, dD_s, \qquad s = o,a, \qquad a = 1,2, \ldots, N \end{aligned} \tag{29}$$

in which

$$\begin{aligned} m_s &= \int_{D_s} \rho_s \, dD_s, \qquad \tilde{\boldsymbol{S}}_s = \int_{D_s} \rho_s \tilde{\boldsymbol{r}}_s \, dD_s, \qquad \boldsymbol{I}_s = \int_{D_s} \rho_s \tilde{\boldsymbol{r}}_s \tilde{\boldsymbol{r}}_s^T \, dD_s \\ \bar{\boldsymbol{\Phi}}_s &= \int_{D_s} \rho_s \boldsymbol{\Phi}_s \, dD_s, \qquad \tilde{\boldsymbol{\Phi}}_s = \int_{D_s} \rho_s \tilde{\boldsymbol{r}}_s \boldsymbol{\Phi}_s \, dD_s \\ \boldsymbol{\Phi}_{os} &= \boldsymbol{\Phi}_o(\boldsymbol{r}_{os}), \qquad \boldsymbol{\Gamma}_s = \nabla \times \boldsymbol{\Phi}_s(\boldsymbol{r}_{os}) \end{aligned} \tag{30}$$

The potential energy is assumed to be due entirely to the elastic deformations and can be written as

$$V = \frac{1}{2}\boldsymbol{q}_o^T \boldsymbol{K}_o \boldsymbol{q}_o + \sum_{a=1}^{N} \frac{1}{2}\boldsymbol{q}_a^T \boldsymbol{K}_a \boldsymbol{q}_a + \sum_{b=1}^{N} \frac{1}{2}\boldsymbol{u}_b^T \boldsymbol{K}_b \boldsymbol{u}_b = \frac{1}{2}\boldsymbol{x}^T \boldsymbol{K} \boldsymbol{x} \tag{31}$$

where $\boldsymbol{K}_o$ and $\boldsymbol{K}_a$ are substructure stiffness matrices, $\boldsymbol{K}_b$ are boundary stiffness matrices due to the action of the springs at the boundary points B, and $\boldsymbol{K}$ is the overall stiffness matrix for the whole structure. Moreover,

$$\boldsymbol{u}_b \cong \boldsymbol{R}_O - (\tilde{\boldsymbol{r}}_{oa} + \boldsymbol{C}_a^T \tilde{\boldsymbol{r}}_{ab} \boldsymbol{C}_a)\boldsymbol{\theta} + \boldsymbol{\Phi}_{oa}\boldsymbol{q}_o + \boldsymbol{C}_a^T \boldsymbol{\Phi}_{ab}\boldsymbol{q}_a \tag{32}$$

represents the displacement vector of point B, in which $\boldsymbol{\Phi}_{ab} = \boldsymbol{\Phi}_a(\boldsymbol{r}_{ab})$. The other quantities on the right side of Eq. (32) were defined earlier. The overall stiffness matrix can be written in the form

$$\boldsymbol{K} = \begin{bmatrix} \kappa_{11} & \kappa_{12} & \kappa_{13} & \kappa_{14}^1 & \kappa_{14}^2 & \cdots & \kappa_{14}^N \\ (\kappa_{12})^T & \kappa_{22} & \kappa_{23} & \kappa_{24}^1 & \kappa_{24}^2 & \cdots & \kappa_{12}^N \\ (\kappa_{13})^T & (\kappa_{23})^T & \boldsymbol{K}_o + \kappa_{33} & \kappa_{34}^1 & \kappa_{34}^2 & \cdots & \kappa_{34}^N \\ (\kappa_{14}^1)^T & (\kappa_{24}^1)^T & (\kappa_{34}^1)^T & \boldsymbol{K}_1 + \kappa_{44}^1 & 0 & \cdots & 0 \\ (\kappa_{14}^2)^T & (\kappa_{24}^2)^T & (\kappa_{34}^2)^T & 0 & \boldsymbol{K}_2 + \kappa_{44}^2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots & \cdot \\ (\kappa_{14}^N)^T & (\kappa_{24}^N)^T & (\kappa_{34}^N)^T & 0 & 0 & \cdots & \boldsymbol{K}_N + \kappa_{44}^N \end{bmatrix} \tag{33}$$

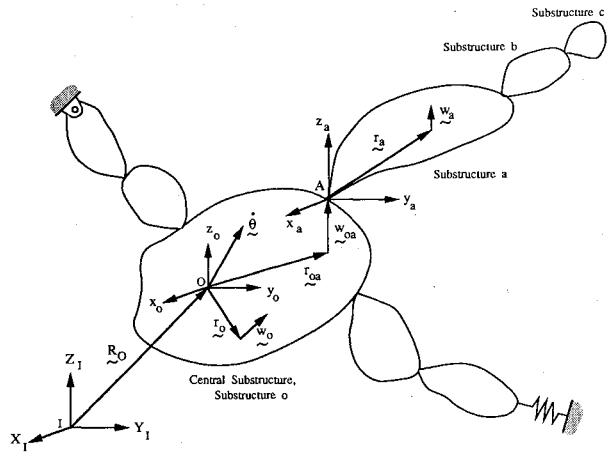

Fig. 4 General flexible multibody model.

in which

$$\kappa_{11} = \sum_{a=1}^{N} k_a, \qquad \kappa_{12} = -\sum_{a=1}^{N} k_a(\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a)$$

$$\kappa_{13} = \sum_{a=1}^{N} k_a \Phi_{oa}$$

$$\kappa_{22} = -\sum_{a=1}^{N} (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a) k_a (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a)$$

$$\kappa_{23} = \sum_{a=1}^{N} (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a) k_a \Phi_{oa}$$

$$\kappa_{33} = \sum_{a=1}^{N} \Phi_{oa}^T k_a \Phi_{oa}, \qquad \kappa_{14}^a = k_a C_a^T \Phi_{ab}$$

$$\kappa_{24}^a = (\tilde{r}_{oa} + C_a^T \tilde{r}_{ab} C_a) k_a C_a^T \Phi_{ab}$$

$$\kappa_{34}^a = \Phi_{oa}^T k_a C_a^T \Phi_{ab}, \qquad \kappa_{44}^a = \Phi_{ab}^T C_a k_a C_a^T \Phi_{ab} \tag{34}$$

As pointed out in the beginning of this section, Lagrange's equations for free vibrations are fully defined by the coefficient matrices in the kinetic energy and potential energy. Indeed, the equations of motion can be written simply as

$$M\ddot{x}(t) + Kx(t) = 0 \tag{35}$$

where M and K are the mass and stiffness matrices for the full structure, as given by Eqs. (28) and (33), respectively. Then, because free vibration of conservative systems is harmonic, $x(t) = e^{i\omega}x$, Eq. (35) yields the eigenvalue problem

$$Kx = \lambda Mx, \qquad \lambda = \omega^2 \tag{36}$$

VI. Convergence Considerations in Substructure Synthesis

It is demonstrated in Ref. 9 that the accuracy of the eigenvalues computed by the variational approach to substructure synthesis can be improved dramatically through the use of quasi-comparison functions instead of mere admissible functions. In this section, we propose to probe deeper into the reason for these superior convergence characteristics.

As shown in Sec. III, if the admissible functions are incapable of satisfying the natural boundary conditions when taken in finite linear combinations, then the computed eigenvalues tend to be very inaccurate. In the example of Fig. 1, the admissible functions given by Eqs. (19) have zero slope at $x = L$. Yet, the natural boundary condition at that point requires a nonzero slope. In theory, one must take an infinite sum of functions with zero slope to produce a finite slope, which explains why the convergence is so poor. As soon as admissible functions with nonzero slope were added to the linear combination, thus permitting the generation of quasi-comparison functions in the form of linear combinations from the functions given by Eqs. (20), convergence improved dramatically. The fact that quasi-comparison functions possess the capability of satisfying all the boundary conditions has profound implications in substructure synthesis.

From Fig. 4, we observe that in the case of flexible multibody systems there are no geometric boundary conditions, except when a substructure has a fixed boundary, so that, for the most part, the only boundary conditions characterizing a substructure are natural. It follows that, according to the Rayleigh-Ritz theory, the admissible functions need satisfy nothing. To be sure, they must be p times differentiable, but this requirement is satisfied routinely by virtually all choices. Hence, in theory, admissible functions corresponding to the modes of the free-free substructure should be a suitable choice. This turns out not to be the case, however. To emphasize the fact that substructure modes are not necessary in the context of substructure synthesis, and in the absence of a better term, in the future we shall refer to substructure modes as trial functions, or shape functions.

In the case of substructures in bending, there are four quantities entering into boundary conditions, namely, displacement, slope, bending moment, and shearing force, where the latter two involve second and third derivatives with respect to the spatial variable, respectively. In component-mode synthesis, geometric compatibility is enforced by means of constraint equations. The substructure synthesis proposed in this paper uses a consistent kinematical procedure, as discussed in Sec. V, ensuring geometric compatibility at boundary points automatically, which is accomplished by defining various sets of body axes so as to guarantee displacement and slope compatibility at boundary points common to any two substructures. This obviates the need for constraint equations enforcing such geometric compatibility. In addition, rigid-body motions are included in the displacement vector of the central substructure o. But this substructure synthesis goes beyond ensuring geometric compatibility. Indeed, through the use of quasi-comparison functions, provisions are also made for matching the bending moment and shearing force at boundary points between any two substructures, thus satisfying natural boundary conditions at these points. In this regard, a distinction must be made between substructure o and substructures $a (a = 1,2, \ldots, N)$. Indeed, for substructure o, it is necessary to make provisions for nonzero displacement, slope, bending moment, and shearing force at the boundary points A. On the other hand, because displacement and slope compatibility are guaranteed automatically by the kinematical procedure, for substructures of type a it is only necessary to make provisions for nonzero bending moment and shearing force at points A. All this is done through a judicious choice of shape functions. For example, linear combinations of free-free shape functions alone do not qualify as quasi-comparison functions for substructure o, because they are characterized by zero bending moment and shearing force at boundary points. However, a set including free-free functions and clamped-clamped functions can qualify as quasi-comparison functions if the set provides for nonzero displacement, slope, bending moment, and shearing force of arbitrary magnitude at all boundary points. This implies that there must be a minimum number of functions in the set. Indeed, if substructure o is a beam in bending, and we confine ourselves to vertical displacements only, there are four arbitrary quantities at each end, for a total of eight. Hence, in addition to one rigid-body translation and one rigid-body rotation, it is necessary to include six shape functions in the set, perhaps three free-free functions and three clamped-clamped functions. This choice of shape functions is far from being unique. Another choice that can prove quite suitable, and one likely to cause initial skepticism, is a set of clamped-free and free-clamped functions. It should be reit-

erated that the preceding shape functions are not modes at all, as there is no conceivable substructure that can be free-free and clamped-clamped or clamped-free and free-clamped at the same time. Going one step further, in the numerical example in the next section we will demonstrate that sine and cosine functions can constitute a suitable set from which to construct quasi-comparison functions.

The fact that a given choice of quasi-comparison functions for the substructures provides both for the satisfaction of geometric compatibility and for the matching of bending moment and shearing force at boundary points does not mean that the natural boundary conditions will be satisfied exactly. Indeed, in rendering the Rayleigh quotient in terms of the energy inner product stationary, the natural boundary conditions will be satisfied approximately, and so will the differential equations. The Rayleigh-Ritz process in conjunction with substructure synthesis will attempt to reduce the error at all points of the structure, regardless of whether they are boundary points or points in the interior of the substructures. In using quasi-comparison functions, instead of mere admissible functions, the process is given the chance to reduce errors at all points of the structure, resulting in superior convergence characteristics.

As mentioned on several occasions, the substructure synthesis is in fact a Rayleigh-Ritz method. The main difference between the substructure synthesis presented here and the classical Rayleigh-Ritz method is that here the admissible functions are local in the sense that they are defined over the domain of a given substructure, whereas in the classical Rayleigh-Ritz method they are global, in the sense that they are defined over the entire structure. Of course, the fact that this substructure synthesis is a Rayleigh-Ritz method has many implications. The most important of these is that most of the theory developed in conjunction with the Rayleigh-Ritz method is valid for the substructure synthesis developed here as well. Having the backing of the Rayleigh-Ritz theory permits us to draw some immediate conclusions concerning the convergence of substructure synthesis. To this end, let us assume that there are s substructures and that the motion of each substructure is represented by a quasi-comparison function in the form of a linear combination of N_i admissible functions ($i = 1,2, \ldots, s$). Hence, the number of degrees of freedom of the system is

$$n = \sum_{i=1}^{s} N_i$$

Then, as the number of admissible functions entering into the comparison functions is increased, we can state that

$$\lim_{n \to \infty} \lambda_r^n = \lambda_r \tag{37}$$

or, the computed eigenvalues converge to the actual eigenvalues. Moreover, they approach the actual eigenvalues from above. Although the Rayleigh-Ritz method does not permit a similar statement concerning the eigenvectors,[7] it is safe to say that, as the number of admissible functions entering into the quasi-comparison functions increases, the error in satisfying the differential equation and the boundary conditions tends to diminish. The rate of convergence depends to some extent on the choice of quasi-comparison functions, and for eigenvalues it tends to be faster than for the eigenvectors. This can be attributed to the stationarity of Rayleigh's quotient,[7] which implies that, if a computed eigenvector differs from the corresponding actual eigenvector by a small quantity of first order in magnitude, the computed eigenvalue differs from the corresponding actual eigenvalue by a small quantity of second order in magnitude.

From the preceding discussion, we conclude that the substructure synthesis presented here is different philosophically from the various versions of component-mode synthesis. In fact, it is closer in nature to the hierarchical finite element method.[11] Indeed, both this substructure synthesis and the hierarchical finite element method describe the motion in terms of local admissible functions. In the first, the local functions are admissible functions capable of yielding quasi-comparison functions defined over full substructures, and in the second, the local functions are polynomials defined over finite elements. Moreover, in substructure synthesis, convergence is achieved by increasing the number of admissible functions entering into the quasi-comparison function for a given substructure, and in the hierarchical finite element method, convergence is achieved by increasing the number and degree of polynomials for a given finite element. The latter is in contrast with the ordinary finite element method, in which convergence is achieved by keeping the number of polynomials constant and refining the finite element mesh.

VII. Numerical Example

The theory just developed is illustrated by means of the structure shown in Fig. 5. The structure consists of three substructures, the central substructure o and two substructures of type a, with the supports being mounted on springs. The central substructure is a uniform beam, and the other two substructures are tapered beams, as shown in Fig. 5.

We propose to describe the motion of the substructures in five different ways. In case 1, we represent the motion of all the substructures by means of mere admissible functions. To this end, we use free-free shape functions for the central substructure and clamped-free shape functions for the remaining two substructures, all functions corresponding to modes for the associated uniform beams. As pointed out earlier, the boundary conditions for the central substructure cannot be satisfied with a finite number of free-free shape functions. Similarly, the boundary conditions at points B_1 and B_2 in the case of substructures of type a cannot be satisfied with a finite number of clamped-free shape functions. Note that the geometric boundary conditions at points A_1 and A_2 are satisfied automatically by the kinematical procedure implied by Eq. (2). In case 2, motion of the central substructure is represented by means of an improved set of admissible functions, and the motion of the remaining two substructures is represented by means of quasi-comparison functions. In particular, for the central substructure, we use a combination of free-free and pinned-pinned shape functions. Although it represents an improvement over case 1, this combination of functions still falls in the class of mere admissible functions, as the bending moment at the boundaries remains zero with a finite number of terms. For the other two substructures, we use a combination of clamped-free and clamped-pinned shape functions, so that this combination of shape functions qualifies as a quasi-comparison function. In cases 3–5, the motion of all the substructures is represented by quasi-comparison functions. In case 3, for the central substructure, we use a combination of free-free and clamped-clamped shape functions, so that both boundary conditions at each end can be satisfied with a finite number of functions. Cases 4 and 5 differ from case 3 in that the motion of the central substructure is represented in case 4 by clamped-free and free-clamped shape functions, and in

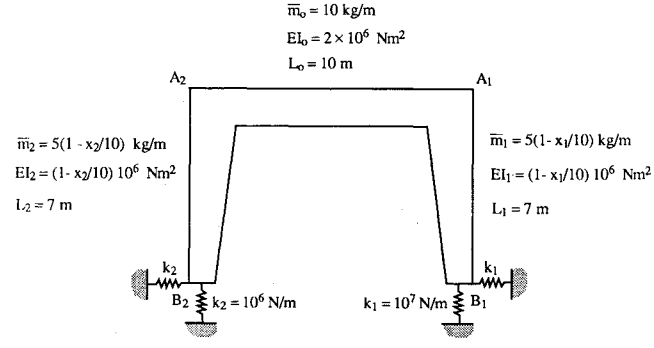

Fig. 5 Example structure.

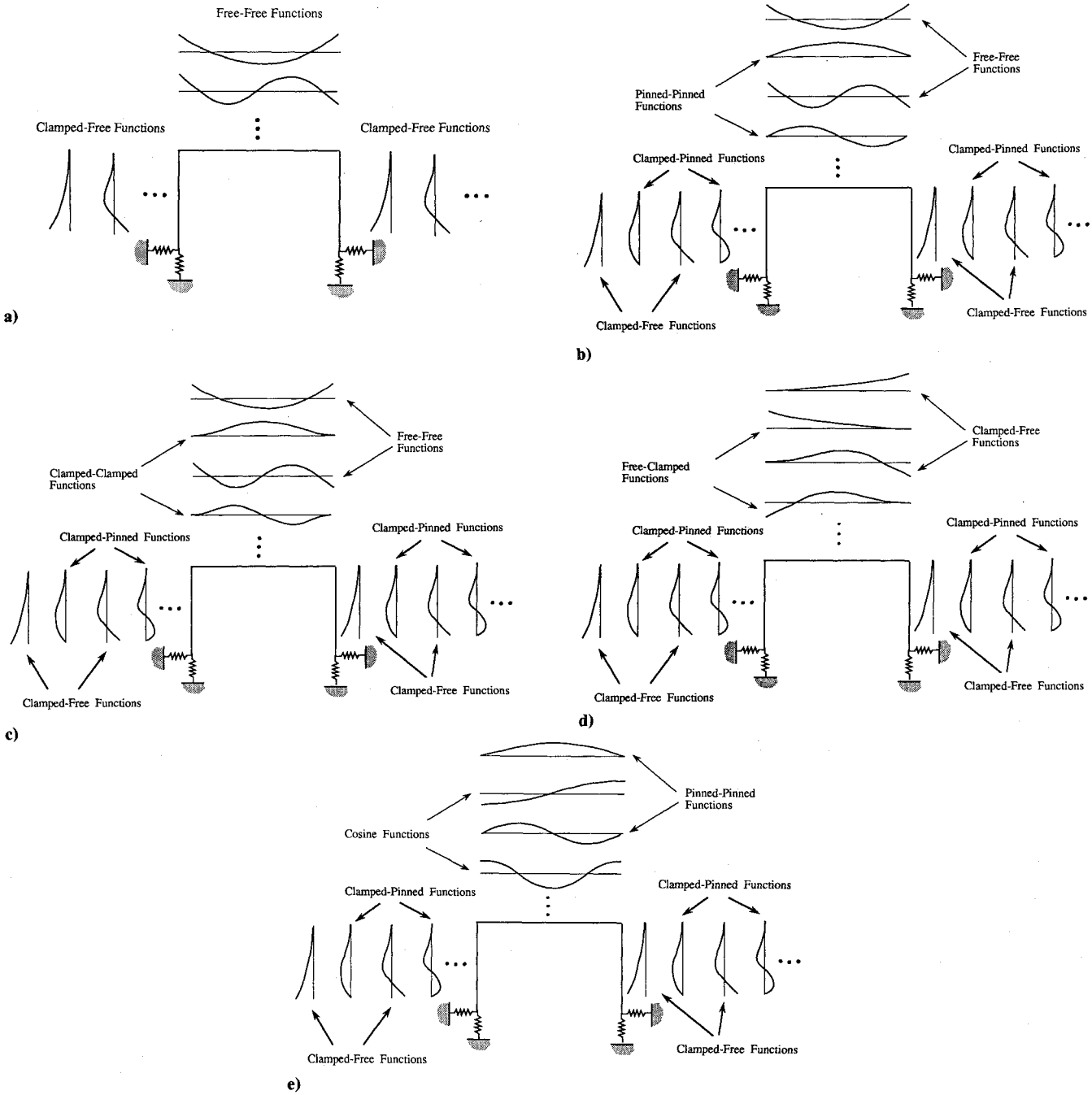

Fig. 6 Shape functions for the example structure.

case 5 by pinned-pinned and cosine shape functions. The motion of the other two substructures is represented in cases 3, 4, and 5 as in case 2. Figures 6a–e show the various shape functions for cases 1–5.

The various types of functions used have closed-form expressions. The free-free shape functions have the expression

$$\phi_i = \cosh\beta_i x + \cos\beta_i x - \sigma_i\,(\sinh\beta_i x + \sin\beta_i x) \qquad (38)$$

The clamped-free, clamped-clamped, and clamped-pinned functions are given by

$$\phi_i = \cosh\beta_i x - \cos\beta_i x - \sigma_i\,(\sinh\beta_i x - \sin\beta_i x) \qquad (39)$$

where the constants β_i and σ_i vary from case to case; their numerical values are given in Ref. 12. On the other hand, the pinned-pinned shape functions have the form

$$\phi_i = \sin(i\pi x/L) \qquad (40)$$

the the cosine functions are given by

$$\phi_i = \cos(i\pi x/L) \qquad (41)$$

Finally, the stiffness matrices for beams in bending have the entries

$$k_{oij} = \int_0^{l_o} EI_o(x)\phi_i''\,\phi_j''\,\mathrm{d}x$$

$$k_{aij} = \int_0^{l_a} EI_a(x)\phi_i''\,\phi_j''\,\mathrm{d}x, \qquad a = 1,2 \qquad (42)$$

where primes denote differentiations with respect to x. The numerical values for the various parameters are shown in Fig. 5.

The eigenvalue problem for the system shown in Fig. 5 was solved for the five cases just described and the three lowest natural frequencies are listed in Tables 3–5. The results are in

Table 3 First natural frequency

DOF	Case 1	Case 2	Case 3	Case 4	Case 5
6	1.75746	1.75746	1.75746	1.69875	1.75745
9	1.57714	1.65474	1.65474	1.47335	1.47530
12	1.56932	1.56538	1.56538	1.47102	1.47084
15	1.53536	1.49632	1.47391	1.47073	1.47077
18	1.53462	1.49632	1.47391	1.47073	1.47077
21	1.52395	1.49632	1.47391	1.47073	1.47073
24	1.52339	1.48726	1.47157	1.47073	1.47073
27	1.51702	1.48138	1.47073	1.47073	1.47073

Table 4 Second natural frequency

DOF	Case 1	Case 2	Case 3	Case 4	Case 5
6	8.59485	8.59485	8.59485	11.62478	8.71827
9	8.29244	8.25873	8.23772	9.76602	8.33865
12	8.26215	8.24909	8.22873	8.22884	8.32848
15	8.25375	8.24867	8.22834	8.21273	8.21143
18	8.24792	8.23106	8.20832	8.21197	8.20475
21	8.24659	8.22109	8.20449	8.20455	8.20475
24	8.24418	8.22108	8.20447	8.20443	8.20475
27	8.24369	8.22107	8.20446	8.20443	8.20444

Table 5 Third natural frequency

DOF	Case 1	Case 2	Case 3	Case 4	Case 5
6	26.86095	26.86095	26.86095	28.98808	26.92279
9	21.78531	21.91468	21.35960	20.37671	20.24966
12	21.30582	21.11908	20.98775	20.18728	19.93381
15	20.66517	20.11777	20.03998	19.94335	19.93298
18	20.61882	20.11582	20.03894	19.93163	19.93137
21	20.46641	20.11493	20.03876	19.93120	19.93120
24	20.45673	20.05149	19.95669	19.93111	19.93111
27	20.37866	20.00927	19.93139	19.93102	19.93102

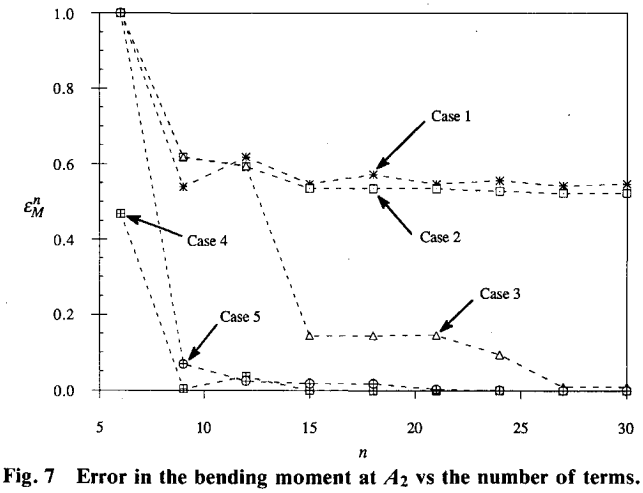

Fig. 7 Error in the bending moment at A_2 vs the number of terms.

agreement with the expectations. In case 1, in which the motion is expressed in terms of one type of admissible functions only for each substructure, the convergence is unsatisfactory. Using a 27-degree-of-freedom model, the computed natural frequencies are relatively far from the actual ones, and improvement with the addition of degrees of freedom is very slow. At this point, there is no indication how many degrees of freedom will be necessary for convergence. In case 2, in which the motion of substructure o is described by means of more suitable admissible functions than in case 1 and the motion of substructures a is represented by means of quasi-comparison functions, the results are significantly better than in case 1, although convergence is still elusive. In cases 3–5, in which the motion of all substructures is represented by means of quasi-comparison functions, convergence is relatively rapid, with the results of case 4 being better than those of cases 3 and 5. Clearly, the results are far superior to those obtained in the first two cases. As pointed out earlier, there is a minimum number of terms necessary before the linear combinations of admissible functions become quasi-comparison functions. Hence, in cases 3–5, the results for small numbers of degrees of freedom are not meaningful.

To verify how well the natural boundary condition at points A is satisfied, we define the error in the bending moment as

$$\epsilon_M^n = M(A-) - M(A+) \tag{43}$$

where $M(A-)$ denotes the bending moment at point A_2 computed from the solution for substructure o and $M(A+)$ is the same quantity corresponding to substructure $a = 2$. Figure 7 shows plots of ϵ_M^n vs n for all five cases just discussed. It is clear that the solutions in terms of quasi-comparison functions are far superior to those in terms of mere admissible functions.

VIII. Summary and Conclusion

This paper addresses the problem of modeling flexible multibody systems by means of a Rayleigh-Ritz based substructure synthesis method.

The use of the variational approach to the Rayleigh-Ritz method to derive the eigenvalue problem has the advantage that the motion can be represented in terms of admissible functions, which tend to be simple functions. Admissible functions need satisfy only the geometric boundary conditions and must be from a complete set in energy. Completeness is a mathematical concept guaranteeing eventual convergence, but giving no indication as to the rate of convergence. In computational structural dynamics, however, the rate of convergence is very important, as the object is to obtain good eigenvalue accuracy with as few degrees of freedom as possible. Quite often, the poor convergence characteristics exhibited by solutions in terms of admissible functions can be traced to the inability to satisfy the natural boundary conditions with a finite number of degrees of freedom.

As pointed out earlier, the Rayleigh-Ritz method as ordinarily used has an implicit flaw, which can impair the convergence characteristics. If the solution of the differential eigenvalue problem is reduced to the problem of rendering Rayleigh's quotient stationary, then an approximate solution can be constructed from the space of admissible functions; such solutions are expressed as linear combinations of trial functions. In general, the trial functions are of the same type, so that the trial functions themselves must be admissible functions. In certain cases, the inability of small linear combinations of admissible functions of the same type to satisfy the natural boundary conditions causes poor convergence. Recognition of this fact prompted the development of a new class of functions, called quasi-comparison functions and characterized by the fact that the members of this class represent linear combinations of admissible functions that act like comparison functions. The quasi-comparison functions were demonstrated to have remarkable convergence characteristics because they are capable of satisfying the natural boundary conditions to any desired degree of accuracy. The question remains as to how to generate quasi-comparison functions. In this regard, it should be mentioned that a common practice in the Rayleigh-Ritz method is to generate admissible functions by solving a related eigenvalue problem. It turns out that in general this is not a good practice, as the process yields admissible functions of a single type, i.e., satisfying a given set of boundary conditions, and such functions tend to be characterized by poor convergence. Indeed, to generate quasi-comparison functions, it is often necessary to combine several types of admissible functions, each type satisfying different boundary conditions.

The concept of quasi-comparison functions is particularly powerful in substructure synthesis, in which a given substructure interacts directly with the adjacent substructures and indirectly with the remaining substructures. This makes it

impossible to specify boundary conditions for a given substructure independently of the other substructures, and hence to define a substructure eigenvalue problem. It follows that substructure eigenfunctions do not really exist, and the same can be said about substructure comparison functions. Moreover, because most boundary conditions tend to be of the natural type, modeling in terms of ordinary admissible functions is unlikely to be sufficiently accurate. Hence, modeling in terms of quasi-comparison functions is virtually the only viable course of action. In approximating the motion of substructures by means of quasi-comparison functions, the boundary conditions between any two substructures can be satisfied to any degree of accuracy with only a finite number of terms. Moreover, the approximate solution tends to be more accurate not only at boundary points but also at points interior to any substructure. As a result, eigensolutions obtained through the use of quasi-comparison functions exhibit superior convergence characteristics compared with eigensolutions obtained by means of ordinary admissible functions. It is very likely that various component-mode synthesis methods are capable of yielding very good results, provided the "modes" used are from the space of quasi-comparison functions.

A numerical example, using as a mathematical model a three-substructure frame supported by springs, demonstrates the various points made above. In particular, the eigenvalues computed by means of quasi-comparison functions are appreciably more accurate than those computed by means of ordinary admissible functions, and they converge much faster. In fact, the convergence of the latter is so slow that it is difficult to state with any degree of conviction that they will converge to the actual eigenvalues. Related to this, the example shows that the use of ordinary admissible functions makes it impossible to provide bending moment balance at the corners of the frame. Similar behavior can be anticipated for the shearing force balance. On the other hand, the solution in terms of quasi-comparison functions permits accurate matching of the bending moment at the corners.

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